

Conic Crease Patterns with Reflecting Rule Lines

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Abstract

The mathematics and algorithms of curved creases remain major challenges in origami science. The goal of this work is to characterize curved-crease origami possible within a particular restricted family of designs, roughly corresponding to the extensive curved-crease designs of the third author (Demaine et al., 2010, 2014; Koschitz, 2014). Specifically, we assume three properties of the design:

1. Every crease is a *quadratic spline*, i.e., decomposes into pieces of conic sections (lines, circles, ellipses, parabolas, or hyperbolas).
2. The *rule segments* (straight line segments on the 3D folded surface) within each face of the crease pattern *converge* to a common point (i.e., pass through that point if extended to infinite lines), specifically, a *focus* of an incident conic. (As in the projective plane, we view parabolas as having one focus at infinity, and lines as having two identical foci at infinity; rule segments passing through a point at infinity means that they are all parallel.)
3. Each crease has a *constant fold angle* all along its length. By an equation of Fuchs and Tabachnikov (2007), this constraint is equivalent to the rule segments *reflecting* through creases, i.e., whenever a rule segment touches the interior of a crease, its reflection through the crease is also (locally) a rule segment (effectively forming a locally flat-foldable vertex at the crease).¹

This family of curved-crease origami designs is natural because, if rule segments converge to a focus of a conic, then the reflected rule segments on the other side of the conic also converge to a focus of that conic. In this way, conics provide a relatively easy way to bridge between pencils of rule segments that converge to various points.

Unfortunately, we show that designs in this family (with more than one crease) rarely “exist” in the sense of properly folding in 3D out of perfect (zero-thickness unstretchable) paper. Following the formalism introduced at last OSME (Demaine et al., 2014), we give a differential equation defining a necessary relation between any two creases connected by rule segments. We use this differential equation to show that rarely can two conic creases be connected by rule segments that satisfy all the conditions above. There are a few exceptions, namely when the two connected conics have identical or reciprocal *eccentricity* (a measure of deviation from circularity). But essentially all other designs, which look like fine crease–rule patterns in 2D, cannot actually fold into 3D.

As an example, the classic Huffman tower of Figure 1 cannot fold with the drawn rule segments because there is an incompatibility between circle and parabola, whose eccentricities are $e = 0$ and $e = 1$ respectively. By contrast, the simpler tower of Figure 2 does properly fold. On the other hand, designs in this family are still interesting because they may (and probably do) fold in 3D with different (nonconverging) rulings. Figure 3 shows a properly folded discrete model of the Huffman tower which, taken to the limit, might give a proper folding of the same crease pattern (but with a different ruling).

References

Erik D. Demaine, Martin L. Demaine, and Duks Koschitz. Reconstructing David Huffman’s legacy in curved-crease folding. In *Origami⁵: Proceedings of the 5th International Conference on Origami in Science, Mathematics and Education*, pages 39–52. A K Peters, Singapore, July 2010.

¹In his early work, the third author called this “refraction”, by analogy to optics (Koschitz, 2014), but geometrically it is reflection.

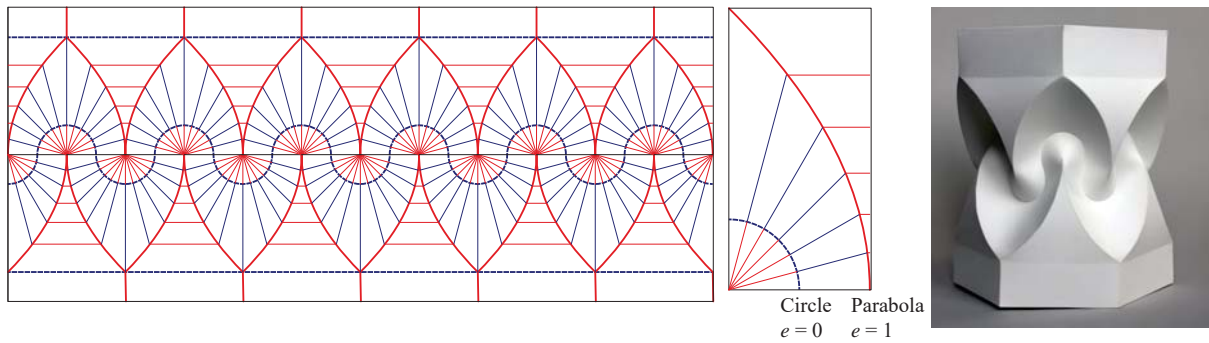


Figure 1: “Hexagonal column with cusps” designed by the third author (Demaine et al., 2010; Koschitz, 2014) drawn with reflecting rule segments passing through the foci of curves. This crease–rule pattern cannot fold because circle (eccentricity $e = 0$) is not compatible with parabola ($e = 1$).

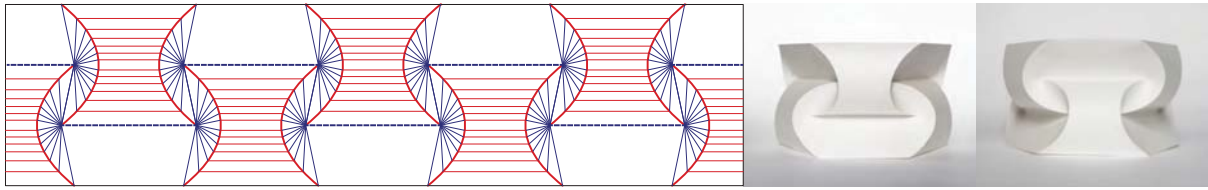


Figure 2: A simple tower using only parabolas, designed by the fourth author in the style of the third author, by tiling a part of his “Arches” design (Koschitz, 2014, Fig. 2.3.12) in a different direction.

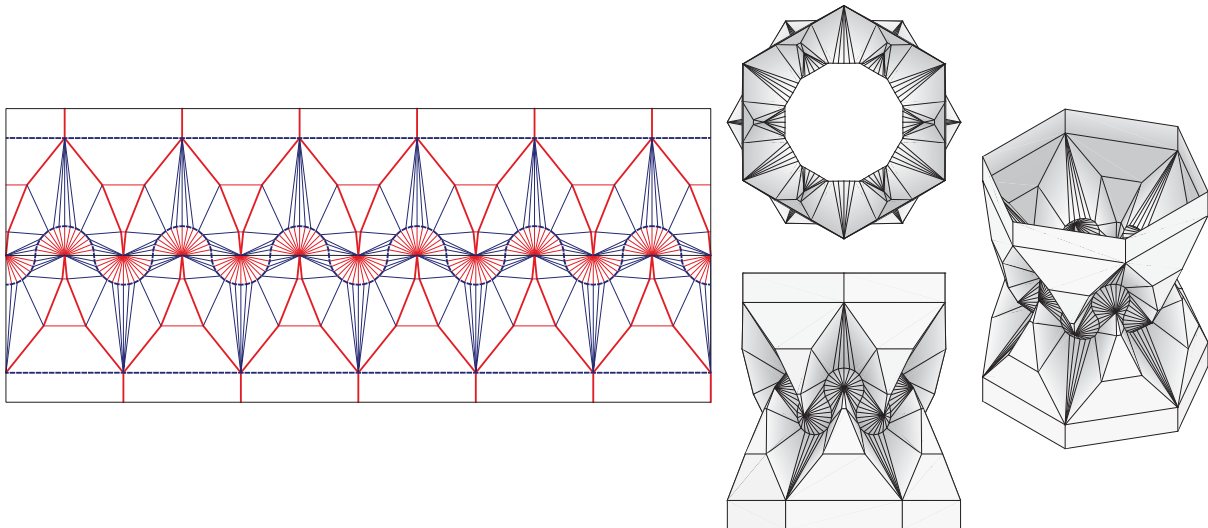


Figure 3: Discrete approximation of Figure 1, suggesting a possible proper ruling pattern (not reflecting through conics).

Erik D. Demaine, Martin L. Demaine, David A. Huffman, Duks Koschitz, and Tomohiro Tachi. Characterization of curved creases and rulings: Design and analysis of lens tessellations. In *Origami⁶: Proceedings of the 6th International Meeting on Origami in Science, Mathematics and Education*, volume 1, pages 209–230. American Mathematical Society, Tokyo, Japan, August 2014.

Dmitry Fuchs and Serge Tabachnikov. Developable surfaces. In *Mathematical Omnibus: Thirty Lectures on Classic Mathematics*, chapter 4. American Mathematical Society, 2007.

Richard Duks Koschitz. *Computational Design with Curved Creases: David Huffman’s Approach to Paperfolding*. PhD thesis, Massachusetts Institute of Technology, September 2014.